

# First-order chiral to non-chiral transition in the angular dependence of the upper critical induction of the Scharnberg-Klemm $p$ -wave pair state

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We calculate the temperature  $T$  and angular  $(\theta, \phi)$  dependence of the upper critical induction  $B_{c2}(\theta, \phi, T)$  for parallel-spin superconductors with an axially symmetric  $p$ -wave pairing interaction pinned to the lattice and a dominant ellipsoidal Fermi surface (FS). For all FS anisotropies, the chiral Scharnberg-Klemm state  $B_{c2}(\theta, \phi, T)$  exceeds that of the chiral Anderson-Brinkman-Morel state, and exhibits a kink at  $\theta = \theta^*(T, \phi)$ , indicative of a first-order transition from its chiral, nodal-direction behavior to its non-chiral, antinodal-direction behavior. Applicability to  $\text{Sr}_2\text{RuO}_4$ ,  $\text{UCoGe}$ , and topological superconductors such as  $\text{Cu}_x\text{Bi}_2\text{Se}_3$  is discussed.

## I. INTRODUCTION

Recently, there has been a great deal of interest in  $p$ -wave superconductivity<sup>1–9,11–25</sup>. The most likely candidate  $p$ -wave superconductors are the ferromagnetic superconductors  $\text{UGe}_2$ ,  $\text{UCoGe}$ , and  $\text{URhGe}$ , which exhibit long-range ferromagnetism well above the superconducting transition temperature  $T_c$ , and the same electrons participate in the ferromagnetism and the superconductivity<sup>1–7</sup>. In  $\text{URhGe}$ , measurements of the temperature  $T$  dependence of the upper critical induction  $B_{c2}(T)$  in the three crystal axis directions was found to fit the Scharnberg-Klemm theory of the  $p$ -wave polar state with completely broken symmetry (CBS)<sup>3,8</sup>, with single-component  $p_z$ -pairing state only along the crystal  $a$ -axis. Subsequent experiments found a reentrant superconducting phase at much higher magnetic field  $H$  strengths, violating the conventional Pauli limit  $B_P = 1.85T_c$  (T/K) by a factor of 20.  $B_{c2}$  in  $\text{UCoGe}$  also violates  $B_P$  by a factor of 20, but its anisotropy suggests that if the superconductivity were  $p$ -wave, it would be more likely to have an axial state form, such as do the chiral Anderson-Brinkman-Morel (ABM) and chiral Scharnberg-Klemm (SK) states<sup>26–29</sup>. Second, there has been an even greater interest in  $\text{Sr}_2\text{RuO}_4$ , as the Knight shift measurements for  $H$  parallel and perpendicular to the layers all showed no temperature  $T$  dependence below  $T_c$ , suggestive of a parallel-spin state<sup>11,12</sup>. However,  $B_{c2}$  experiments on that material were shown to be strongly Pauli limited for  $B \perp \hat{c}$ <sup>13–17,19,30</sup>, and scanning tunneling microscopy experiments showed strong evidence for a nodeless gap<sup>18</sup>, although with cylindrical Fermi surfaces (FSs), this might be consistent with an axial  $p$ -wave state. Third, there has been a large recent interest in topological insulators, in the hope that they might become chiral  $p$ -wave superconductors with doping, applied pressure, or proximity coupling<sup>20–25</sup>. Initial  $B_{c2}(T)$  measurements on  $\text{Cu}_x\text{Bi}_2\text{Se}_3$  were consistent with a  $p$ -wave polar state for  $H$  both parallel and perpendicular to the layers<sup>23,26</sup>. However, scanning tunneling microscopy (STM) experiments established that  $\text{Cu}_x\text{Bi}_2\text{Se}_3$  has an isotropic gap strongly suggestive of an  $s$ -wave order parameter (OP)<sup>25</sup>,

and that isotropic  $s$ -wave OP was respectively proximity-induced up to 7 K and 50 K into  $\text{Bi}_2\text{Se}_3$  layers deposited atop the  $c$ -axes of the layered low- $T_c$  and high- $T_c$  superconductors,  $2H\text{-NbSe}_2$  and  $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ <sup>32,33</sup>, consistent with  $s$ -wave substrate crystal OPs in the  $c$ -axis direction of both of those layered superconductors<sup>31,34–36</sup>. However, still undiscovered topological superconductors might have axial  $p$ -wave OP symmetry.

Previously, we generalized the microscopic calculation of  $B_{c2}(T)$  for the  $p$ -wave polar state pinned to a crystal lattice direction to extend its validity to a superconductor with a dominant ellipsoidal FS and  $B$  in an arbitrary direction,  $B = B(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ , with respect to the crystal lattice, in order to provide a sound theoretical basis for a more sensitive probe of the actual OP in orthorhombic materials such as  $\text{URhGe}$ . Here we use the same technique to construct a theory of the full angular dependence of  $B_{c2}(\theta, \phi, T)$  for the ABM and SK states, in order to identify the symmetry of the OP in  $\text{UCoGe}$ ,  $\text{Sr}_2\text{RuO}_4$ , and other candidate materials. Since  $\text{UCoGe}$  is orthorhombic, the ellipsoidal FS model is the best that can be made without additional features such as magnetic pairing fluctuation effects and  $B$  dependencies of the pairing interactions<sup>37</sup>, or  $B$ -dependent interactions<sup>38</sup>, *etc.* For tetragonal  $\text{Sr}_2\text{RuO}_4$ , the lack of any detectable ferromagnetism strongly suggests weak coupling interactions, but there are three barrel shaped FSs, and the STM experiments strongly suggest nearly equal isotropic gaps on each<sup>18</sup>. Although one could envision a scenario in which one FS dominated  $B_{c2}(0^\circ, \phi, T)$  and another dominated  $B_{c2}(90^\circ, \phi, T)$ , since the latter is of primary interest, it suffices to consider only one FS. Moreover, as the  $k_z$  dispersion of those bands is sufficient to avoid dimensional crossover effects in  $B_{c2}(90^\circ, \phi, T)$  measurements<sup>15,31,39,40</sup>, an ellipsoid of uniaxial anisotropy is sufficient to examine  $B_{c2}$  measurements for all  $B$  directions with high accuracy<sup>31</sup>. As anticipated earlier, for a parallel-spin pairing interaction of the form  $V(\hat{k}, \hat{k}') = 3V_0(\hat{k}_1\hat{k}'_1 + \hat{k}_2\hat{k}'_2)$ , one would expect  $B_{c2}(\theta, \phi, T)$  to be given by the SK state<sup>9,26</sup>. Although a favorite pair state for  $\text{Sr}_2\text{RuO}_4$  has the form  $\hat{z}(\hat{k}_1 + i\hat{k}_2)$ , where the  $d$ -vector  $\hat{z}$  corresponds to the antiparallel-spin

state in the lattice representation, we shall here assume that the spins are parallel<sup>26</sup>, and will include Pauli limiting effects subsequently<sup>30</sup>. Here we present detailed calculations of the  $B_{c2}(\theta, \phi, T)$  for both the ABM and SK states on a single ellipsoidal FS.

We assume weak coupling for a clean homogeneous type-II parallel-spin  $p$ -wave superconductor with effective Hamiltonian<sup>9,26</sup>,

$$\mathcal{H} = \sum_{\mathbf{k}, \sigma=\pm} a_{\mathbf{k}, \sigma}^\dagger [\epsilon(\mathbf{k} - e\mathbf{A}) - \mu] a_{\mathbf{k}, \sigma} + \frac{1}{2} \sum_{\mathbf{k}, \mathbf{k}', \sigma} a_{\mathbf{k}', \sigma}^\dagger a_{\mathbf{k}, \sigma}^\dagger V(\hat{\mathbf{k}}, \hat{\mathbf{k}}') a_{\mathbf{k}, \sigma} a_{\mathbf{k}', \sigma}, \quad (1)$$

$$V(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \frac{3}{2} V_0 \sum_{\sigma'=\pm} f_{\sigma'}(\hat{\mathbf{k}}) \hat{\mathbf{d}}_{\sigma'} \cdot \hat{\mathbf{d}}_{\sigma'}^* f_{\sigma'}^*(\hat{\mathbf{k}}'), \quad (2)$$

where we assume parallel-spin pairing with  $\hat{\mathbf{d}}_{\sigma'} = \hat{\mathbf{x}} + i\sigma'\hat{\mathbf{y}}$  and  $f_{\sigma'}(\hat{\mathbf{k}}) = (\hat{k}_1 + i\sigma'\hat{k}_2)$  from the degenerate  $\Gamma_3^-$  and  $\Gamma_4^-$  tetragonal point group representations<sup>10</sup>,  $e$  is the electronic charge,  $\mu$  is the chemical potential, the unit wave vectors  $\hat{k}_i$  were previously defined on an ellipsoidal FS<sup>9</sup>, and we set  $\hbar = k_B = 1$ . For non-ferromagnetic candidate  $p$ -wave superconductors, the upper critical induction  $B_{c2} = \mu_0 H_{c2}$ , where  $H_{c2}$  is the upper critical field. After performing the Klemm-Clem (KC) transformations<sup>41</sup> that map the ellipsoidal FS onto a spherical one and then rotate the transformed induction to the new  $\tilde{z}$  axis direction, the transformed linear gap equation becomes

$$\bar{\Delta}(\tilde{\mathbf{R}}, \hat{\mathbf{k}}) = T \sum_{\omega_n} \frac{N(0)}{2} \int d\Omega_{\tilde{\mathbf{k}}} \tilde{V}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') \int_0^\infty d\xi_{\tilde{\mathbf{k}}} \times e^{-2\xi_{\tilde{\mathbf{k}}}'|\omega_n|} e^{-i\xi_{\tilde{\mathbf{k}}}' v_F \hat{\mathbf{k}}' \cdot \tilde{\mathbf{R}}} \bar{\Delta}(\tilde{\mathbf{R}}, \hat{\mathbf{k}}'), \quad (3)$$

where  $\bar{\Delta}$  is the transformed  $\Delta$  amplitude without the gauge phases<sup>9</sup>,  $N(0) = mk_F/(2\pi^2)$  is the density of states per spin at the chemical potential  $\mu$  for an effectively isotropic metal with a geometric mean mass  $m = (m_1 m_2 m_3)^{1/3}$ , effective Fermi wave vector  $k_F = \sqrt{2m\mu}$ , effective Fermi velocity  $v_F = k_F/m$ , and  $\tilde{\mathbf{R}} = -i\alpha \tilde{\mathbf{R}} - 2e\tilde{\mathbf{A}}(\tilde{\mathbf{R}})$ , where  $\alpha(\theta, \phi) = \sqrt{m_3} \sqrt{\cos^2 \theta + \gamma^{-2}(\phi) \sin^2 \theta}$ ,  $\bar{m}_i = m_i/m$ , and  $\gamma^2(\phi) = \frac{m_3}{m_1 \cos^2 \phi + m_2 \sin^2 \phi}$  is the ellipsoidal anisotropy function<sup>9</sup>. The KC transformations change  $V(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$  in Eq. (2) to

$$\tilde{V}(\hat{\mathbf{k}}, \hat{\mathbf{k}}') = \frac{3}{2} V_0 \sum_{\sigma=\pm} \tilde{f}_\sigma(\hat{\mathbf{k}}) \tilde{f}_\sigma^*(\hat{\mathbf{k}}') \quad (4)$$

where  $\tilde{f}_\sigma(\hat{\mathbf{k}}) = \hat{k}_1 + i\sigma(\hat{k}_2 \cos \theta' + \hat{k}_3 \sin \theta')$ ,  $\cos \theta' = \sqrt{m_3} \cos \theta / \alpha$ , etc.<sup>9</sup> From the form of  $\tilde{V}(\hat{\mathbf{k}}, \hat{\mathbf{k}}')$ ,  $\bar{\Delta}(\tilde{\mathbf{R}}, \hat{\mathbf{k}}) = \sum_{\sigma=\pm} \bar{\Delta}_\sigma(\tilde{\mathbf{R}}) \tilde{f}_\sigma(\hat{\mathbf{k}})$ , we expand the  $\bar{\Delta}_\sigma(\tilde{\mathbf{R}})$  in terms of the harmonic oscillator eigenfunctions  $|n(\tilde{\mathbf{R}})\rangle$ ,  $\bar{\Delta}_\sigma(\tilde{\mathbf{R}}) = \sum_{n=0}^\infty a_n^\sigma |n(\tilde{\mathbf{R}})\rangle$ , perform the integrals over the  $\hat{k}_i$  vari-

ables in the linearized gap equation, and obtain this double recursion relation for the  $a_n^{(\pm)}$ ,

$$\begin{aligned} a_n^{(\pm)} &= \left( \frac{1}{2} (1 + \cos^2 \theta') a_n^{(\pm)} + \frac{1}{2} \sin^2 \theta' a_n^{(\mp)} \right) \alpha_n^{(a)} \\ &+ \frac{1}{2} \sin^2 \theta' (a_n^{(\pm)} - a_n^{(\mp)}) \alpha_n^{(p)} \\ &+ \left( \frac{1}{4} \sin^2 \theta' a_{n+2}^{(\pm)} + \frac{1}{4} (1 \pm \cos \theta')^2 a_{n+2}^{(\mp)} \right) \beta_n \\ &+ \left( \frac{1}{4} \sin^2 \theta' a_{n-2}^{(\pm)} + \frac{1}{4} (1 \mp \cos \theta')^2 a_{n-2}^{(\mp)} \right) \beta_{n-2} \end{aligned} \quad (5)$$

where

$$\begin{aligned} \alpha_n^{(p,a)} &= \pi T \sum_{\omega_n} \int_0^\pi d\theta_{\tilde{\mathbf{k}}} \sin \theta_{\tilde{\mathbf{k}}} \left( 3 \cos^2 \theta_{\tilde{\mathbf{k}}}', \frac{3}{2} \sin^2 \theta_{\tilde{\mathbf{k}}}' \right) \\ &\times \int_0^\infty d\xi_{\tilde{\mathbf{k}}} e^{-2\xi_{\tilde{\mathbf{k}}}'|\omega_n|} e^{-\eta_{\tilde{\mathbf{k}}}'/2} L_n(\eta_{\tilde{\mathbf{k}}}'), \quad (6) \\ \beta_n &= \pi T \sum_{\omega_n} \int_0^\pi d\theta_{\tilde{\mathbf{k}}} \frac{3}{2} \sin^3 \theta_{\tilde{\mathbf{k}}} \int_0^\infty d\xi_{\tilde{\mathbf{k}}} e^{-2\xi_{\tilde{\mathbf{k}}}'|\omega_n|} \\ &\times e^{-\eta_{\tilde{\mathbf{k}}}'/2} (-\eta_{\tilde{\mathbf{k}}}') L_n^{(2)}(\eta_{\tilde{\mathbf{k}}}') [(n+1)(n+2)]^{-1/2} \end{aligned} \quad (7)$$

where

$$\eta_{\tilde{\mathbf{k}}} = eB\alpha(\theta, \phi) v_F^2 \xi_{\tilde{\mathbf{k}}}^2 \sin^2 \theta_{\tilde{\mathbf{k}}}', \quad (8)$$

$t = T/T_c$ ,  $T_c = (2e^C \omega_0 / \pi) \exp(-1/N(0)V_0)$ ,  $\omega_0$  is a characteristic pairing cutoff frequency,  $C \approx 0.5772$  is Euler's constant, and  $L_n(z)$  and  $L_n^{(2)}(z)$  are a Laguerre and an associated Laguerre polynomial, respectively<sup>9,26</sup>.

For the chiral ABM state, the decoupled  $a_n^{(\pm)}$  each satisfy  $a_n^{(\pm)} D_n = \Gamma_n a_{n+2}^{(\pm)} + \Gamma_{n-2} a_{n-2}^{(\pm)}$ , where  $D_n = 1 - \frac{1}{2}(1 + \cos^2 \theta') \alpha_n^{(a)} - \frac{1}{2} \sin^2 \theta' \alpha_n^{(p)}$  and  $\Gamma_n = \frac{1}{4} \sin^2 \theta' \beta_n$ . Solving this recursion relation, we obtain the continued fraction expression from which  $B_{c2}(\theta, \phi, t)$  for the ABM state is obtained numerically,

$$D_0 - \frac{\Gamma_0^2}{D_2 - \frac{\Gamma_2^2}{D_4 - \dots}} = 0. \quad (9)$$

As for the polar/CBS state<sup>9</sup>, one iteration is accurate to a few percent, but four or five iterations are needed for the accuracy necessary to observe the interesting effects.

The results for the reduced  $b_{c2}(\theta, t)$  for a parallel-spin superconductor in the  $p$ -wave ABM state with a dominant spherical  $\gamma^2(\phi) = 1$  FS are shown in Fig. 1. In Fig. 1(a), the curves for  $\theta = 0^\circ(\mathbf{b} \parallel \hat{\mathbf{c}})$  (nodal direction) to  $90^\circ(\mathbf{b} \perp \hat{\mathbf{c}})$  (antinodeal direction) are shown in increments of  $10^\circ$ . The result for the nodal direction ( $\theta = 0^\circ$ ) was obtained previously<sup>26</sup>. Just below  $T_c$ ,  $b_{c2}(\theta, \phi, t) \propto [m_3 \cos^2 \theta + 2\gamma^{-2}(\phi) \sin^2 \theta]^{-1/2}$ , where the factor 2 arises from the ABM order parameter (OP) anisotropy. In order to distinguish which part of the overall  $b_{c2}(\theta, t)$  anisotropy that is attributable solely to the order parameter anisotropy, in Fig. 1(b), those Fig. 1(a) results scaled to have the same slope at  $t = 1$  are presented. Nothing unusual is evident from these spherical

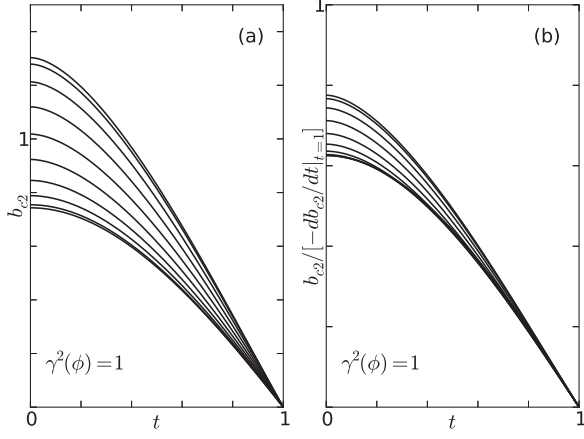


FIG. 1: (a) Reduced  $b_{c2}$  versus  $t = T/T_c$  for the chiral ABM state [Eq. (9)] at  $\theta$  values from  $0^\circ$  ( $\mathbf{H} \parallel \hat{\mathbf{c}}$ , bottom) to  $90^\circ$  ( $\mathbf{H} \perp \hat{\mathbf{c}}$ , top), in increments of  $10^\circ$  for a spherical FS. (b) Same curves normalized to have the same slopes at  $T_c$ .

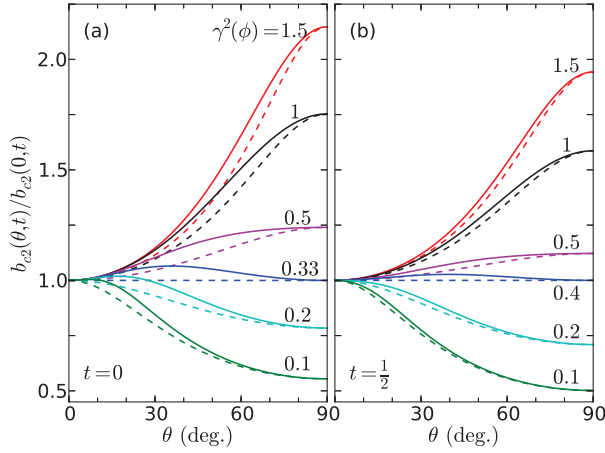


FIG. 2: (color online) Reduced  $b_{c2}$  versus  $\theta$  for the chiral ABM state [Eq. (9)] at the indicated effective mass anisotropy  $\gamma^2(\phi)$  values (solid) and the effective mass angular fits [Eq. (10), dashed] at  $t = 0$  (a) and  $t = 1/2$  (b).

FS curves, and they are smooth and increase monotonically with increasing  $\theta$ .

However, we also studied the role of ellipsoidal (or uniaxial) FS anisotropy. In Fig. 2, we chose fixed FS anisotropy values  $\gamma^2(\phi)$  ranging from 0.1 to 1.5 and plotted in Figs. 1 (a,b) at  $t = 0$  and  $\frac{1}{2}$ , respectively. The solid curves are evaluated from Eq. (9). The dashed curves are the conventional “effective mass” anisotropy  $b_{\text{eff}}(\theta, t)$  forms obtained by fitting the calculated  $b_{c2}(0^\circ, t)$  and  $b_{c2}(90^\circ, t)$ ,

$$b_{\text{eff}}(\theta, t) = [\cos^2 \theta / b_{c2}^2(0^\circ, t) + \sin^2 \theta / b_{c2}^2(90^\circ, t)]^{-1/2} \quad (10)$$

We note that  $b_{c2}(\theta, t)$  exhibits an unusual  $\theta$  dependence, with a peak in at  $\theta^*$  for  $\gamma^2(\phi) < \frac{1}{2}$  that is distinctly different than the conventional  $b_{c2}$  maxima at  $\theta = 0^\circ$  or  $90^\circ$ .

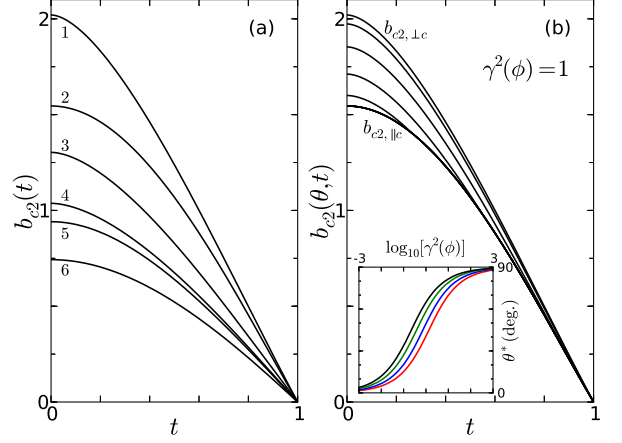


FIG. 3: (color online) (a)  $b_{c2}(t)$  for the antinodal SK state (1), nodal SK state (2), antinodal ABM state (3), s-wave state absent of Pauli limiting (4), planar nodal (CBS) state (5), and nodal ABM state (6) on a spherical FS. (b) Reduced  $b_{c2}(t)$  for the chiral SK state at  $\theta$  values from  $0^\circ$  ( $\mathbf{B} \parallel \hat{\mathbf{c}}$ , bottom) to  $90^\circ$  ( $\mathbf{B} \perp \hat{\mathbf{c}}$ , top), in increments of  $10^\circ$  for a spherical FS. The  $\theta = 0^\circ, 10^\circ, 20^\circ, 30^\circ$  and  $40^\circ$  are indistinguishable on this scale. Inset: Plots of the kink angle  $\theta^*$  versus  $\log_{10}[\gamma^2(\phi)]$  from top to bottom for  $t = \frac{3}{4}$  (black),  $\frac{1}{2}$  (green),  $\frac{1}{4}$  (blue), 0 (red).

Such anomalous double peaks at unconventional  $\theta$  values satisfying  $0 < \theta^* < 90^\circ$ , and by reflection symmetry about  $90^\circ$ , also for  $90^\circ < \theta^* < 180^\circ$ , were predicted earlier for the polar state pinned to the lattice<sup>9</sup>. However, in that case, the anomalous double peaks were predicted to occur for  $\lambda(t) > \gamma^2(\phi) > 3$ , with maximal  $\lambda(t)$  values for finite  $t$ . Since the anomalous behavior is unlikely to be relevant to either  $\text{Sr}_2\text{RuO}_4$  or  $\text{UCoGe}$ , for which  $\gamma^2 \gg 1$ , for brevity, the  $\lambda'(t)$  curve defining the lower limit of the range of  $\theta^*$  for  $\lambda'(t) < \gamma^2(\phi) < \frac{1}{2}$  will be presented elsewhere<sup>30</sup>.

The much more interesting chiral axial  $p$ -wave state is the SK state. We note that it is chiral as long as  $\Delta^{(+)} \neq \Delta^{(-)}$ , or  $a_n^{(+)} \neq a_n^{(-)}$  for at least one relevant  $n$  value<sup>9</sup>. It is easy to see that for  $\theta' = 0$ , Eq. 5) reduces for  $a_n^{(+)} \neq 0$  to  $[1 - \alpha_n^{(a)}][1 - \alpha_{n+2}^{(a)}] = \beta_n^2$ , which for  $a_n^{(+)} \neq 0$  is the expression for the SK state with  $\mathbf{B}$  in the nodal direction<sup>26</sup>, whereas for  $\theta' = \pi/2$ , it reduces for  $a_n^{(+)} \neq a_n^{(-)}$  to  $\alpha_n^{(p)} = 1$ , the expression for the SK state with  $\mathbf{B}$  in the antinodal (polar state) direction<sup>26</sup>.

However, for a general  $\theta'$ ,  $a_n^{(+)} \neq a_n^{(-)}$ , Eq. (5) is a double recursion relation in the six harmonic oscillator amplitudes,  $a_n^{(\pm)}$ ,  $a_{n+1}^{(\pm)}$  and  $a_{n-2}^{(\pm)}$ , which requires further analysis to write the exact solution. We first write  $\Psi_n^{(\pm)} = \frac{1}{2}(a_n^{(+)} \pm a_n^{(-)})$ ,  $D_n^{(+)} = 1 - \alpha_n^{(a)}$ ,  $D_n^{(-)} = 1 - \alpha_n^{(a)} \cos^2 \theta' - \alpha_n^{(p)} \sin^2 \theta'$ , and construct  $\phi_n^{(\pm)} = \cos \theta' D_n^{(+)} \Psi_n^{(+)} \pm D_n^{(-)} \Psi_n^{(-)}$ . After letting  $n \rightarrow n + 2$  in the expression for  $\phi_n^{(-)}$ , we obtain two equations for  $\Psi_n^{(+)}$  and  $\Psi_{n+2}^{(+)}$  in terms of  $\Psi_n^{(-)}$  and  $\Psi_{n+2}^{(-)}$ . Using these equa-

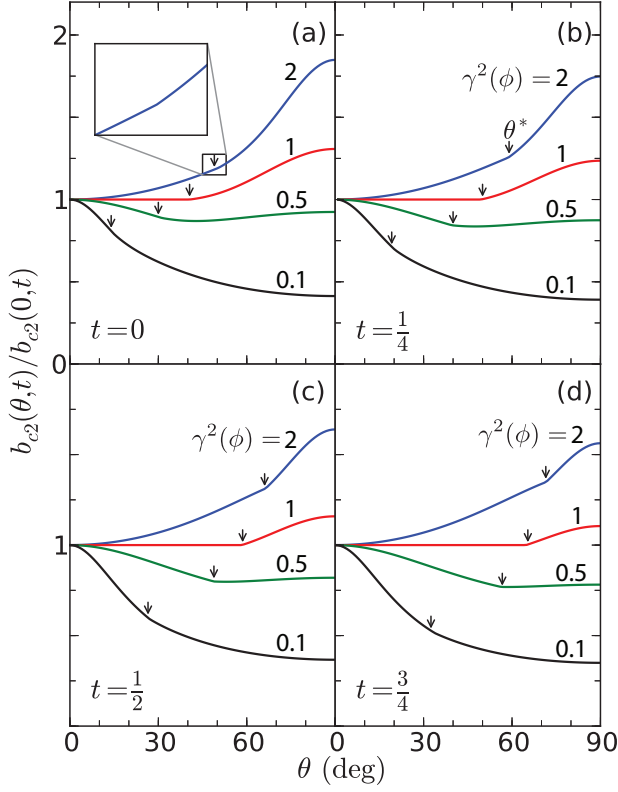


FIG. 4: (color online) Reduced upper critical induction  $b_{c2}$  versus  $\theta$  for the chiral SK state for  $\gamma^2(\phi) = 2$  (blue, top), 1 (red), 0.5 (green), and 0.1 (black) at  $t = 0$  (a),  $\frac{1}{4}$  (b),  $\frac{1}{2}$  (c), and  $\frac{3}{4}$  (d). The arrows indicate kinks in  $b_{c2}(\theta)$  at  $\theta^*$ , signifying first-order transitions from the chiral SK state ( $\theta < \theta^*$ ) to the non-chiral antinodal SK (or polar) state  $b_{c2}(t)$  curve ( $\theta > \theta^*$ ).

tions to eliminate  $\Psi_n^{(+)}$  and  $\Psi_{n+2}^{(+)}$  in favor of  $\Psi_n^{(-)}$  and  $\Psi_{n+2}^{(-)}$ , letting  $n \rightarrow n - 2$  in the expression for  $\Psi_{n+2}^{(-)}$ , and equating that with the other expression for  $\Psi_n^{(-)}$ , we obtain the simple recursion relation for the  $\Psi_n^{(-)}$ ,  $A_n \Psi_{n+2}^{(-)} + B_n \Psi_n^{(-)} + C_n \Psi_{n-2}^{(-)} = 0$ , the solution of which may be expressed in the continued fraction equation,

$$B_0 - \frac{A_0 C_0}{B_2 - \frac{A_2 C_2}{B_4 - \dots}} = 0, \quad (11)$$

where  $B_n = B_n^{(+)} - B_n^{(-)}$ ,  $A_n = E_{n-2} \beta_n [\cos^2 \theta' D_{n+2}^{(+)} - D_{n+2}^{(-)}]$ ,  $B_n^{(+)} = D_n^{(-)} [E_n D_{n-2}^{(+)} + E_{n-2} D_{n+2}^{(+)}]$ ,  $B_n^{(-)} = \cos^2 \theta' [\beta_n^2 E_{n-2} + \beta_{n-2}^2 E_n]$ ,  $C_n = \beta_{n-2} E_n [\beta_{n-2} \cos^2 \theta' D_{n-2}^{(+)} - D_{n-2}^{(-)}]$ , and  $E_n = D_n^{(+)} D_{n+2}^{(+)} - \beta_n^2$ . As for the polar/CBS state and the ABM state, one iteration is accurate to a few percent, but four or five iterations are necessary to display the most important features of this work. We also eliminated  $\Psi_n^{(-)}$  and  $\Psi_{n+2}^{(-)}$  in favor of  $\Psi_n^{(+)}$  and  $\Psi_{n+2}^{(+)}$ , but the  $b_{c2}(\theta, \phi, t)$  values calculated from the resulting continued fraction equation were always lower than those calculated from Eq. (11).

In Fig. 3(a), we plotted the reduced  $b_{c2}(t)$  for the nodal and antinodal directions of the ABM, SK, and polar/CBS states, along with that [curve (4)] of a conventional  $s$ -wave superconductor without any Pauli limiting effects, all for a spherical FS. The antinodal directions of the polar state and SK states both have  $b_{c2}(t)$  curves described by curve (1), and the nodal direction of the SK state  $b_{c2}(t)$  follows curve (2), as found previously<sup>26</sup>. Curve (3) is the new  $b_{c2}(t)$  curve for the antinodal direction of the ABM state. Curves (5) and (6) describe the planar nodal polar/CBS state direction and the nodal direction of the ABM state, as also found previously<sup>8</sup>. We note that the SK state  $b_{c2}(\theta, \phi, t)$  is larger for all field directions than is the ABM state  $b_{c2}(\theta, \phi, t)$ , as the second chiral component of the OP allows for the state to be superconducting at larger applied field strengths. In Fig. 3(b), the  $t$  dependence of  $b_{c2}(\theta, t)$  is illustrated for  $\theta = 0^\circ$  ( $\mathbf{b} \parallel \hat{\mathbf{c}}$ ) (bottom) to  $\theta = 90^\circ$  ( $\mathbf{b} \perp \hat{\mathbf{c}}$ ) (top), in increments of  $10^\circ$ . Surprisingly, the curves for  $\theta = 0^\circ, 10^\circ, 20^\circ, 30^\circ$  and  $40^\circ$  are remarkably close to one another, and appear to cross at finite  $t$  values! This is an indication of a chiral to non-chiral transition for  $\theta \geq 40^\circ$  at various  $t$  values, as the vortices just below  $b_{c2}$  appear to lock onto the nodal direction for  $\theta \leq 40^\circ$ , but for  $\theta > 40^\circ$  unlock from that direction, and favor the non-chiral antinodal (or polar) state direction. Similar behavior was predicted recently for the vortex structure in the mixed state of a chiral ABM state model of  $\text{Sr}_2\text{RuO}_4$ <sup>19</sup>.

To investigate this surprising feature in more detail, in Fig. 4 we show the  $\theta$  dependence of  $b_{c2}(\theta, \phi, t)$  at the effective mass anisotropy values  $\gamma^2(\phi) = 0.1, 0.5, 1$ , and  $2$ , at  $t = 0, \frac{1}{4}, \frac{1}{2}$ , and  $\frac{3}{4}$ . In every case, there is a kink in  $b_{c2}(\theta)$  at  $\theta = \theta^*$  for fixed  $\phi$  and  $t$ , which we interpret as evidence for a first-order phase transition from a chiral to non-chiral state. Although these kinks are easiest to see for small  $\gamma^2$  values, and  $\text{Sr}_2\text{RuO}_4$  has  $\gamma^2 > 10^3$ , our high-accuracy solutions of Eq. (11) allow us to determine  $\theta^*[\gamma^2(\phi), t]$  with great precision. In the inset to Fig. 3(b), we plotted  $\theta^*$  in degrees versus  $\ln_{10}[\gamma^2(\phi)]$  from  $-3$  to  $3$  at the reduced  $t$  values  $0, \frac{1}{2}, \frac{1}{4}$ , and  $\frac{3}{4}$ . Thus, if  $\text{Sr}_2\text{RuO}_4$  were a chiral  $p$ -wave parallel-spin superconductor as often purported, then one ought to observe a first order chiral to non-chiral transition for  $\theta \approx 90^\circ$ , nearly parallel to the layers. It is therefore quite interesting to note that some evidence for this sort of behavior may have already been observed in very recent  $H_{c2}(T)$  measurement on  $\text{Sr}_2\text{RuO}_4$ <sup>17</sup>. However, a cautionary note that is that  $b_{c2}(90^\circ, \phi, t)$  appears to be strongly Pauli limited<sup>13–16</sup>, and more details of such and other fits using this FS model will soon become available<sup>30</sup>.

With regards to the ferromagnetic superconductor UCoGe, the ferromagnetism in the  $c$ -axis direction allows for an axial-type parallel-spin  $p$ -wave pairing interaction, most likely mediated by ferromagnetic exchange interactions, in the  $ab$  plane. However, at large applied fields along the  $b$ -axis direction, not only does  $B_{c2,b}(0)$  exceed the Pauli limit by a factor of at least 20, but the very strange behavior of  $B_{c2,b}(T)$ , including prelim-



inary evidence for an  $S$ -shaped curve, strongly suggests something akin to a reentrant superconducting phase overlapping the low-field phase, which would be similar to the two phases of URhGe. Fitting such behavior will require significant modifications to the theory, such as by including ferromagnetic fluctuations<sup>37</sup>, field-dependent interactions<sup>38</sup>, different FS shapes,<sup>43,44</sup> and two ferromagnetically-split FSs, which modifications are currently under study<sup>45</sup>. Although an axial  $p$ -wave topological superconductor is presently elusive, this theory could be useful to identify a future candidate material.

In summary, we have studied the two most-common versions of an axially-symmetric  $p$ -wave pair state, the Anderson-Brinkman-Morel (ABM) and Scharnberg-Klemm (SK) states. For all induction  $\mathbf{B}$  directions and temperatures  $T$ , the reduced (dimensionless) the SK state  $B_{c2}(\theta, \phi, t)$  exceeds that of the ABM state. Surprisingly, for  $0 \leq \theta \leq \theta^*$ , the only  $\theta$ -dependence of  $B_{c2}(\theta, \phi, t)$  arises from effective mass anisotropy, but then

$B_{c2}(\theta)$  exhibits a kink at  $\theta^*[t, \gamma^2(\phi)]$ . Hence, it appears that there are two basic states evident in  $b_{c2}(\theta, \phi, t)$ : the nodal, chiral SK state for  $-\theta^* \leq \theta \leq \theta^*$ , and the antinodal, non-chiral polar state for  $\theta^* \geq \theta \geq -\theta^*$ .

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